Asymptotic behaviour of a porous medium equation with fractional diffusion

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Abstract

We consider a porous medium equation with nonlocal diffusion effects given by an inverse fractional Laplacian operator. In a previous paper we have found mass-preserving, nonnegative weak solutions of the equation satisfying energy estimates. The equation is posed in the whole space \mathbb{R}^n . Here we establish the large-time behaviour. We first find selfsimilar nonnegative solutions by solving an elliptic obstacle problem with fractional Laplacian for the pair pressure-density, which we call obstacle Barenblatt solutions. The theory for elliptic fractional problems with obstacles has been recently established. We then use entropy methods to show that the asymptotic behavior of general finite-mass solutions is described after renormalization by these special solutions, which represent a surprising variation of the Barenblatt profiles of the standard porous medium model.

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1 Introduction

In the paper [11] we have introduced a model of nonlinear diffusion with nonlocal effects given by the system

(1.1)
$$u_t = \nabla \cdot (u \nabla p), \quad p = \mathcal{K}(u).$$

Here, u is a function of the variables (x,t) to be thought of as a density or concentration, and therefore nonnegative, while p is the pressure, which is related to u via a linear operator \mathcal{K} , which we assume to be the inverse of a fractional Laplacian, $\mathcal{K} = (-\Delta)^{-s}$, 0 < s < 1. The problem is posed for $x \in \mathbb{R}^n$, $n \ge 1$, and t > 0, and we give initial conditions

$$(1.2) u(x,0) = u_0(x), \quad x \in \mathbb{R}^n,$$

where u_0 is a nonnegative and integrable function in \mathbb{R}^n with compact support or fast decay at infinity. Motivation and related ideas for such a problem are given in [11], where the existence of a weak solution is established for a suitable class of initial data that includes all functions u_0 with the above properties. Besides, a number of the most important properties is proved, like the energy estimates, the bounds in the L^p spaces and the property of finite propagation that says that compactly supported data produce solutions whose support is compact for all positive times. However, the uniqueness of weak solutions is a pending open problem but for the case of one space dimension, [4]. Comparison theorems, a crucial tool in parabolic equations are only available under special circumstances (i.e., for so-called true super- or sub-solutions), as shown in [11]. We will also need the results of [9], where we show that solutions with integrable data are actually bounded (so-called L^1 - L^{∞} effect) and furthermore that bounded solutions are C^{α} continuous with Hölder exponent depending on the equation and local Hölder constant depending on the L^{∞} norm of the solution.

In this paper we study the asymptotic behaviour of such weak solutions. First, we introduce the rescaled flow and then we prove that the stable configurations of this modified equation are just the solutions of an obstacle problem with fractional Laplacian, as studied by Athanasopoulos, Caffarelli, Salsa and Silvestre, [2] and [7]. This connection, described in Section 3, is quite surprising. In the original variables such equilibrium solution translates into a selfsimilar solution that we propose to name "fractional Barenblatt solution" after the analogy with the porous medium case, though the analytical properties are quite different, see Theorem 3.1. In Sections 4, 5 we prove convergence of the rescaled flow to equilibrium by using a certain

"Boltzmann entropy" technique; in the original variables this means that the asymptotic behaviour of finite mass, compactly supported solutions of our problem is given by the family of fractional Barenblatt solutions constructed from the stationary fractional obstacle problem. Uniqueness of the asymptotic profiles is proved in Section 6 and the proof of the main result, Theorem 4.1, is completed in Section 7. A final Section 8 introduces the open question of spectral gap.

Notation. We will use the notation $L_s = (-\Delta)^s$ with 0 < s < 1 for the fractional powers of the Laplace operator defined by Fourier transform on the Schwartz class of smooth functions in \mathbb{R}^n and extended in a natural way to functions in the Sobolev space $H^{2s}(\mathbb{R}^n)$. Technical reasons of paper [11] imply that in one space dimension the restriction s < 1/2 will be observed. The inverse operator is denoted by $\mathcal{K}_s = (-\Delta)^{-s}$ and can be realized by convolution

$$\mathcal{K}_s = K_s \star u, \qquad K_s(x) = c(n,s)|x|^{2s-n}.$$

see [17]. \mathcal{K}_s is a positive self-adjoint operator. We will write $\mathcal{H}_s = \mathcal{K}_s^{1/2}$ which has kernel $K_{s/2}$. The subscript s will be omitted when s is fixed and known. For functions that depend on x and t, convolution is applied for every fixed t with respect to the space variables. We then use the abbreviated notation $u(t) = u(\cdot, t)$.

RELATED WORKS. Some papers have appeared since 2008 dealing with similar equations. As closely related, we mention the recent research note by Biler et al. [3] where a more general version of our model is proposed, and existence of solutions, decay estimates and selfsimilar solutions are announced. Full details are still to be provided. At least in the latter topic, their method is different and interesting. The asymptotic behaviour is not tackled.

2 Existence result and basic estimates

Definition. We say that u is a weak solution of equation (1.1) in $Q_T = \mathbb{R}^n \times (0, T)$ with initial data $u_0 \in L^1(\mathbb{R}^n)$ if $u \in L^1(Q_T)$, $\mathcal{K}(u) \in W^{1,1}_{loc}(Q_T)$, and $u \nabla \mathcal{K}(u) \in L^1(Q_T)$, and the identity

(2.1)
$$\iint u \left(\phi_t - \nabla \mathcal{K}(u) \cdot \nabla \phi\right) dx dt + \int u_0(x) \phi(x, 0) dx = 0$$

holds for all continuous test functions ϕ in Q_T such that $\nabla_x \phi$ is continuous, and ϕ has compact support in the space variable and vanishes near t = T. The following result is proved in [11]. When $T = \infty$ we write Q instead of Q_T .

Proposition 2.1 Let $u_0 \in L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$, $u_0 \geq 0$, and such that

(2.2)
$$u_0(x) \le A e^{-a|x|}$$
 for some $A, a > 0$.

Then there exists a weak solution u of Equation (1.1) with initial data u_0 . Besides, $u \in L^{\infty}(0, \infty : L^1(\mathbb{R}^n)), u \in L^{\infty}(Q), \nabla \mathcal{H}(u) \in L^2(Q)$. For all t > 0 we have

(2.3)
$$\int_{\mathbb{R}^n} u(x,t) dx = \int_{\mathbb{R}^n} u_0(x) dx,$$

and $||u(t)||_{\infty} \leq ||u_0||_{\infty}$. The solution decays exponentially as $|x| \to \infty$. The first energy inequality holds in the form

(2.4)
$$\int_0^{t_1} \int_{\mathbb{R}^n} |\nabla \mathcal{H}u|^2 dx dt + \int_{\mathbb{R}^n} u(t_1) \log(u(t_1)) dx \le \int_{\mathbb{R}^n} u_0 \log(u_0) dx ,$$

while the second says that for all $0 < t_1 < t_2 < \infty$

(2.5)
$$\int_{t_1}^{t_2} \int_{\mathbb{P}^n} u |\nabla \mathcal{K} u|^2 dx dt + \frac{1}{2} \int_{\mathbb{P}^n} |\mathcal{H} u(t_2)|^2 dx \le \frac{1}{2} \int_{\mathbb{P}^n} |\mathcal{H} (u(t_1))|^2 dx .$$

Properties of the constructed solutions. Here are some of the most useful

- Conservation of mass

(2.6)
$$\frac{d}{dt} \int u(x,t) \, dx = 0.$$

- L^p estimates. We also prove that the L^p norm of the solutions, 1 , does not increase in time.
- Conservation of sign: $u_0 \ge 0$ implies that $u(t) \ge 0$ for all times.
- Persistence of strict positivity: At every point x_0 where $u(x_0, t_0) > 0$ for some time t_0 we have $u(x_0, t) > 0$ for all later times $t > t_0$.
- Finite propagation. Solutions with compact support: One of the most important features of the porous medium equation and other related degenerate parabolic equations is the property of finite propagation, whereby compactly supported initial data $u_0(x)$ give rise to solutions u(x,t) that have the same property for all positive times, i.e., the support of $u(\cdot,t)$ is contained in a ball $B_{R(t)}(0)$ for all t>0.

Proposition 2.2 [11] Assume that u is a bounded solution, $0 \le u \le L$, of equation (1.1) with $\mathcal{K} = (-\Delta)^{-s}$ with 0 < s < 1 (0 < s < 1/2 if n = 1), as constructed in Theorem 2.1. Assume that u_0 has compact support. Then $u(\cdot,t)$ is compactly supported for all t > 0. More precisely, if 0 < s < 1/2 and u_0 is below the "parabola-like" function

$$(2.7) U_0(x) = a(|x| - b)^2,$$

for some a, b > 0, with support in the ball $B_b(0)$, then there is C(n, s, L, a, T) large enough, such that

$$(2.8) u(x,t) \le a(Ct - (|x| - b))^2$$

for $x \in \mathbb{R}$ and 0 < t < T.

- A standard comparison result for parabolic equations does not seem to work. This is one of the main technical difficulties in the study of this equation. In fact, we find special situations where some comparison holds by using so-called true super- and subsolutions.
- Next, we state the L^1 to L^{∞} result that is proved in the forthcoming paper [9].

Proposition 2.3 Let u be a weak solution of Problem (1.1)–(1.2) with $u_0 \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$, and u_0 decreases exponentially as $x \to \infty$. Then there exists a positive constant C such that for every t > 0

(2.9)
$$\sup_{x \in \mathbb{R}^N} |u(x,t)| \le C t^{-\alpha} ||u_0||_{L^1(\mathbb{R}^N)}^{\gamma}$$

with $\alpha = n/(n+2-2s)$, $\gamma = (2-2s)/((n+2-2s))$. The constant C depends only on n and s.

- **Regularity.** Bounded solutions are C^{α} smooth for some $\alpha > 0$: The Hölder exponent depends on n and s and the local Hölder constant depends also on the L^{∞} norm of the solution. This is proved in [9].

3 Large time behavior. Rescaled flow

We begin here the study of the large time behavior. As a first step in the analysis we will introduce self-similar variables, typical in porous medium theory, leading to a rescaled evolution equation. In the analysis of the steady states of that flow we will discover the solutions of an elliptic obstacle problem with fractional diffusion. This surprising connection is the main novelty of the paper, and the basis for the analysis of stabilization to be done in subsequent sections. The precise obstacle result is carefully stated in Theorem 3.1.

We take a weak solution $u \geq 0$ with integrable and bounded initial data, as constructed in [11]. Inspired by the asymptotics of the standard porous medium equation, we define the rescaled flow through the transformation

(3.1)
$$u(x,t) = (1+t)^{-\alpha}v(x(1+t)^{-\beta},\tau)$$

with new time $\tau = \log(1+t)$. We also put $y = x (1+t)^{-\beta}$ as rescaled space variable. In order to cancel the factors including t in a explicit way, we get the condition on the exponents

$$(3.2) \alpha + (2 - 2s)\beta = 1,$$

where we use the homogeneity of K in the form

(3.3)
$$(\mathcal{K}u)(x,t) = (1+t)^{-\alpha+2s\beta}(\mathcal{K}v)(y,\tau).$$

Since we also want conservation of (finite) mass, we must put $\alpha = n\beta$. We get the precise value of the exponents:

(3.4)
$$\beta = 1/(n+2-2s), \quad \alpha = n/(n+2-2s).$$

We recall that 0 < s < 1 so that $\beta \in (1/(n+2), 1/n)$. The first value is the standard porous medium case. In this way, we arrive at the nonlinear, nonlocal Fokker-Planck equation

(3.5)
$$v_{\tau} = \nabla_{y} \cdot \left(v \left(\nabla_{y} \mathcal{K}(v) + \beta y \right) \right)$$

with β as given above. Note that this formula implies a transformation for the pressure of the form

(3.6)

$$p(u)(x,t) = (1+t)^{-\sigma}p(v)(x(1+t)^{-\beta},\tau), \text{ with } \sigma = \alpha - 2s\beta = 1 - 2\beta = \frac{n-2s}{n+2-2s} < 1.$$

We are not making much use of this last formula.

• Equilibrium states for the rescaled flow

We want to find stationary solutions of the rescaled equation (3.5), i.e., solutions V(y) of the system

(3.7)
$$\nabla_y \cdot (V \nabla_y (P + a|y|^2)) = 0, \quad P = \mathcal{K}(V).$$

where $a = \beta/2$, and β defined just above. Since we are looking for asymptotic profiles of the standard solutions of (1.1) we also want $V \ge 0$ and integrable. The simplest possibility is integrating once and getting the radial version

(3.8)
$$V \nabla_y (P + a|y|^2) = 0, \quad P = \mathcal{K}(V), \quad V \ge 0.$$

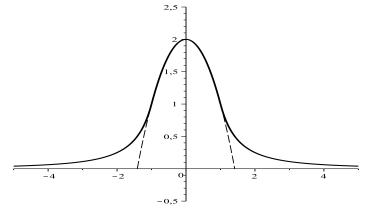
The first equation gives an alternative choice that reminds us of the complementary formulation of the obstacle problems, [12], [5].

• Obstacle problem. Barenblatt solutions of new type

Indeed, if we solve the obstacle problem with fractional Laplacian we will obtain a unique solution P(y) of the problem:

(3.9)
$$P \ge \Phi, \quad V = (-\Delta)^s P \ge 0;$$
 either $P = \Phi$ or $V = 0$.

with 0 < s < 1. In our present application we have to choose as obstacle $\Phi = C - a|y|^2$, where C is any positive constant and $a = \beta/2$. For uniqueness we also need the condition $P \to 0$ as $|y| \to \infty$. The theory is developed in the papers [2, 7], the solution of this obstacle problem is unique and belongs to the space H^{-s} with pressure in H^s . The solutions have the following regularity: $P \in C^{1,s}(\mathbb{R}^n)$ and $V \in C^{1-s}(\mathbb{R}^n)$. The figure represents an approximate plot of P(y).



Note that for $C \leq 0$ the solution is trivial, P = 0, V = 0, hence we choose C > 0. We also note that the pressure is defined but for a constant, so that we may take without loss of generality C = 0 and take as pressure $\widehat{P} = P - C$ instead of P. But then $P \to 0$ implies that $\widehat{P} \to -C$ as $|y| \to \infty$, so we get a one parameter family of stationary profiles that we denote $V_C(y)$. In any case, if we consider the free boundary location, $|x| = R(C, \sigma, n)$, which is the boundary of the contact set of the obstacle problem, $C = \{|y| \leq R(C)\}$, then R tends to 0 as $C \to 0$, and R tends to ∞ as $C \to \infty$. Let us summarize the results of this section.

Theorem 3.1 For every C > 0 there exists a unique solution $P = P_C(y) \in C^{1,s}$ of the obstacle problem (3.9) with $\Phi(y) = C - a|y|^2$ and $P \to 0$ as $|y| \to \infty$. It is radially symmetric as a function of y and we have the scaling law

(3.10)
$$P_C(y) = C P_1(y/C^{1/2}).$$

Moreover, $V_C = (-\Delta)^s P_C$ satisfies the scaling relation $V_C(y) = C^{1-s}V_1(y/C^{1/2})$ and $V_1(y)$ has compact support. Finally, if we put $a = \beta/2 > 0$, then

(3.11)
$$U_C(x,t) = (1+t)^{-\alpha} V_C(x(1+t)^{-\beta})$$

is a weak solution of Equation (1.1).

Note that $U_C(x,t)$ is a weak solution of Equation (1.1) in the sense we have defined. It lives for t > -1 and it takes at the initial time t = -1 a Dirac delta $M \delta(x)$ as initial data, i.e., it is the source-type or Barenblatt solution for this problem, and it has a profile $V_C \geq 0$ that has compact support. In the sequel we call these solutions the family of fractional Barenblatt solutions.

Some further observations: an easy calculation gives the realationship

(3.12)
$$M = \int U_C(x,t) dx = \int V_C(y,\tau) dy = c(n,s) C^{(n+2-2s)/2}$$

(compare with the PME case in [19], page 23). Clearly, $V \geq 0$ is supported in the contact set of the obstacle problem, $C = \{|y| \leq R(C)\}$. The radius R is smaller than the intersection of the parabola Φ with the axis $R_1 = (C/a)^{1/2}$. On the other hand, the pressure $P_C(|y|)$ is always positive and decays to zero as $|y| \to \infty$ according to fractional potential theory, [15, 17].

4 Entropy estimates for the rescaled problem

Our next problem is now to prove that these profiles are attractors for the rescaled flow that we have introduced. In the next sections we prove the main result of this paper.

Theorem 4.1 Let $u(x,t) \ge 0$ be a weak solution of Problem (1.1)–(1.2) with bounded and integrable initial data such that $u_0 \ge 0$ has finite entropy in the sense defined in formula (4.4). Let $v(y,\tau)$ be the corresponding rescaled solution. As $\tau \to \infty$ we have

$$(4.1) v(\cdot,\tau) \to V_C(y) in L^1(\mathbb{R}^n) and also in L^\infty(\mathbb{R}^n).$$

The constant C is determined by the rule of mass equality: $\int_{\mathbb{R}^n} v(y,\tau) dy = \int_{\mathbb{R}^n} V_C(y) dy$. In terms of u this translates into (4.2)

$$u(x,t) - U_C(x,t) \to 0$$
 in $L^1(\mathbb{R}^n)$, $t^{\alpha}|u(x,t) - U_C(x,t)| \to 0$ uniformly in x ,

both limits taken as $t \to \infty$.

As a first step in the proof, we review the estimates of Section 2 in order to adapt them to the rescaled problem. There is no problem is reproving mass conservation or positivity for the rescaled flow, we leave it to the reader checking both facts. By shifting a bit the origin of time, there is no loss of generality in assuming that the entropies mentioned below are finite even at t = 0 even if they were not assumed to be so. The values of α and β are those fixed in (3.4).

• The first energy estimate becomes (recall the notation $\mathcal{H} = \mathcal{K}^{1/2}$)

(4.3)
$$\frac{d}{d\tau} \int v(y,\tau) \log v(y,\tau) dy = -\int |\nabla \mathcal{H}v|^2 dy - \beta \int \nabla v \cdot y$$
$$= -\int |\nabla \mathcal{H}v|^2 dy + \alpha \int v.$$

• However, the second energy estimate has an essential change. We need to define the entropy of the rescaled flow as

(4.4)
$$\mathcal{E}(v(\tau)) := \frac{1}{2} \int_{\mathbb{R}^n} (v \, \mathcal{K}(v) + \beta |y|^2 v) \, dy$$

The entropy contains two terms. The first is

$$\mathcal{E}_1(v(\tau)) := \int_{\mathbb{D}^n} v \, \mathcal{K}(v) \, dy = \int_{\mathbb{D}^n} |\mathcal{H}v|^2 \, dy,$$

hence positive. The second is the moment $\mathcal{E}_2(v(\tau)) = M_2(v(\tau)) := \int |y|^2 v \, dy$, also positive. By differentiation we get

(4.5)
$$\frac{d}{d\tau}\mathcal{E}(v) = -\mathcal{I}(v), \qquad \mathcal{I}(v) := \int \left| \nabla (\mathcal{K}v + \frac{\beta}{2}|y|^2) \right|^2 v dy.$$

We call $\mathcal{I}(v)$ the entropy dissipation of the rescaled solution v. This is formal, the rigorous formula says that whenever the initial entropy is finite, then $\mathcal{E}(v(\tau))$ is uniformly bounded for all $\tau > 0$, $\mathcal{I}(v)$ is integrable in $(0, \infty)$, and

(4.6)
$$\mathcal{E}(v(\tau)) + \int_0^{\tau} \int_{\mathbb{R}^n} \left| \nabla (\mathcal{K}v + \frac{\beta}{2} |y|^2) \right|^2 v dy dt \le \mathcal{E}(v_0).$$

The bound on $\mathcal{E}(v(\tau))$ implies a bound for the energy $\mathcal{E}_1(v(\tau))$ and for the moment $M_2(v(\tau))$ in $L^{\infty}(0,\infty)$. We also conclude that

(4.7)
$$\int_0^\infty d\tau \int_{\mathbb{R}^n} dy \, v \, |\nabla (\mathcal{K}v + (\beta/2)|y|^2)|^2 < \infty.$$

This estimate shows that the integral $\mathcal{I}(v(\tau))$ converges to zero as $\tau \to 0$ in some kind of time average, which is a basic fact in the asymptotic analysis.

Other consequences: (1) For the original variable: the fact that $\int v(y,\tau)|y|^2dy$ is bounded uniformly in time implies in terms of the original variable u that

$$\int u(x,t) \, x^2 dx \le C \, t^{2\beta}.$$

(2) The fact that $\int |\mathcal{H}(v)|^2 dy$ is bounded uniformly in time implies that

$$\int |\mathcal{H}(u)|^2 dy \le C t^{2s\beta - \alpha}.$$

5 Asymptotic behavior. Zero entropy dissipation

We take a solution u(x,t) under the same assumptions on the initial data, we rescale it as in the previous sections to obtain $v(y,\tau)$, and we put $\int v_0(y) dy = M > 0$. Now the idea is to let $\tau \to \infty$ in the rescaled flow equation (3.5). Since the entropy (4.4) is nonnegative and goes down, there is a limit

$$E_* = \lim_{t \to \infty} \mathcal{E}(\tau) \ge 0.$$

Moreover, v is bounded in L_y^1 uniformly in τ , and also $\int |\nabla \mathcal{H}(v)|^2 dy$ is uniformly bounded, in time, therefore we have that $v(\cdot,\tau)$ is a compact family and there is a subsequence $v(\cdot,\tau_j)$ with $\tau_j \to \infty$ that converges in L_y^1 and almost everywhere to a limit $v_* \geq 0$.

In this section we prove the following intermediate result:

Lemma 5.1 the mass of v_* is the same as the mass of $v(\cdot, \tau)$, $\int v_*(y) dy = M$ (hence, in particular, v_* is not trivial). The functions $v_*(y,\tau)$ and $w(y,\tau) = \mathcal{K}v_* + \frac{\beta}{2}|y|^2$ are continuous and $w(\cdot,\tau)$ is constant on every connected component of the set where v_* is not zero,

Proof. We want to use compactness of the family of time-translates

$$v_i(y,\tau) = v(y,\tau + \tau_i), \qquad \tau_i \to \infty.$$

We observe that the L^{∞} estimate of Theorem (2.3) with sharp exponents implies that $v(y,\tau)$ is uniformly bounded in time (for all times $\tau \geq 1$), hence the family v_j is uniformly bounded. Moreover, bounded families of solutions are uniformly equicontinuous (Hölder continuous), and this applies both to u and v. This was proved as part of the local regularity of paper [9]. Therefore, the convergence $v_j \to v_*$ takes place locally uniformly in $Q = \mathbb{R}^n \times (0, \infty)$.

Next, we prove the conservation of mass. This a consequence of the uniform estimate of $v(\cdot,\tau)$ in $L^{\infty}(\mathbb{R}^n)$ for all large t, the local compactness of the solutions, and finally the uniform boundedness of the moment $\int v(y,\tau) |y|^2 dy$. Another immediate consequence is that the *lim inf* of the entropy $\mathcal{E}(v(\tau_i))$ is equal or larger that $M_2(v_*)$.

We also have $\mathcal{H}(v) \in L^2_y$ uniformly in τ as well as $\nabla \mathcal{H}(v)$ in $L^2_{y,\tau}$. Since $\mathcal{K}(v) = \mathcal{H}(\mathcal{H}(v))$ and $\nabla \mathcal{K}(v) = \nabla \mathcal{H}(\mathcal{H}(v)) = \mathcal{H}(\nabla(\mathcal{H}v))$, we derive from the bound for $\nabla \mathcal{H}v$ in $L^2_{y,\tau}$ estimates for $\mathcal{K}(v)$. We recall that $\nabla \mathcal{H}(v)$ is a "derivative of order 1-s of v", and since v is bounded, $v \in L^\infty_{y,\tau}$, we conclude that $v \in L^2_t H^{1-s}_{x,loc}$. By potential theory, it is then clear that $\mathcal{K}(v) \in L^2_t H^{1+s}_x$.

All of this can be used in passing to the limit in the term $\iint v(\nabla \mathcal{K}v)\nabla \phi \,dxdt$ as follows: we have the convergence of v_j in $C([0,T]:L^2(B_R))$ together with the weak convergence of $p_j = \mathcal{K}v_j$ and ∇p_j . In this way, we find that v_* is a weak solution of the rescaled equation (3.5).

Finally, we want to prove that this solution is stationary. For that we pass to the limit in the entropy dissipation bound that says that for every h > 0 fixed

$$\lim_{j \to \infty} \int_0^h \int \left| \nabla (\mathcal{K}v_j(y, \tau) + \frac{\beta}{2} |y|^2) \right|^2 v_j(y, \tau) \, dy \, d\tau = 0.$$

By the strong and weak convergence results that we have, it follows that

(5.1)
$$\mathcal{I}(v_*) = \int_0^h \int \left| \nabla (\mathcal{K}v_*(y,\tau) + \frac{\beta}{2} |y|^2) \right|^2 v_*(y,\tau), dy d\tau = 0.$$

This implies that if $w(y,\tau) = \mathcal{K}v_* + \frac{\beta}{2}|y|^2$ then $v_*|\nabla w|^2 = 0$ a.e. in Q. It follows that for almost every time the continuous function w must be a constant in space in every connected component of the set where the continuous function v_* is not zero, $w(\cdot,\tau) = K(\tau)$. By continuity of w, this happens indeed for all times, but the space constant might still vary continuously in time.

The determination of the space constant, its independence of time, and finally the precise form of v_* will allow us the unique identification of the limit as a solution of the Barenblatt obstacle problem with the same mass $\int v_*(y) dy = M$. This is done in the next section. and the proof of the asymptotic theorem is completed in Section 7.

6 Uniqueness for the asymptotic limit

We will prove here that the vanishing of the entropy dissipation \mathcal{I} characterizes the stationary solutions of the rescaled flow. We use the notations u and x instead of v and y in this "elliptic section". We also put $\beta=1$ for simplicity. Thus, we assume that $\widetilde{u}(x)\geq 0$ is continuous in \mathbb{R}^n , that $\widetilde{p}=(-\Delta)^{-s}\widetilde{u}$ is also continuous, and that we have the basic identity

$$\int |\nabla(\widetilde{p} - Q)|^2 \widetilde{u} \, dx = 0,$$

where $Q(x) = -|x|^2/2$. Then

Theorem 6.1 There is a $c_0 > 0$ such that \widetilde{p} equals $P(c_0)$, the solution of the fractional obstacle problem with obstacle $\Phi = Q + c_0$. Moreover, $P(c_0) \to 0$ as $x \to \infty$.

We proceed via a series of lemmas. We write $P_0 = P(c_0)$.

Lemma 6.2 \widetilde{p} attains its maximum only at x = 0

Proof. By the maximum principle, \tilde{p} cannot attain its maximum at a point where $\tilde{u} = 0$, since then $(-\Delta)^s \tilde{p} = 0$ there, and this is impossible at a maximum of \tilde{p} according

to the integral formula that describes the operator $(-\Delta)^s$ (unless \widetilde{p} is constant, which is not the case). Hence, \widetilde{p} must attain its maximum at a point where $\widetilde{u} > 0$. Besides, at the maximum x_0 we have $\nabla \widetilde{p}(x_0) = 0$, hence $\nabla Q(x_0) = 0$, and this means that $x_0 = 0$.

Corollary 6.3 (i) We have $\widetilde{u}(0) > 0$. (ii) $\widetilde{p} = Q + c_0$ in a neighborhood Ω of x = 0.

Lemma 6.4 Let P_0 is the solution of the fractional obstacle problem for data $\Phi = Q + c_0$, then $P_0 - Q$ is strictly convex.

Proof. Let e be a unit vector and let h > 0. We have

$$\frac{1}{2} \left[P_0(x+he) + P_0(x-he) \right] \ge \frac{1}{2} \left[2c_0 - \frac{1}{2} \left(|x+he|^2 + |x-he|^2 \right) \right] = c_0 - \frac{1}{2} |x|^2 - \frac{1}{2} h^2 = Q(x) + c_0 - \frac{1}{2} h^2.$$

Therefore, since P_0 is a supersolution of $(-\Delta)^s p(x) \geq 0$, also the function

$$\gamma(x,h) = \frac{1}{2} \left[P_0(x + he) + P_0(x - he) \right] + \frac{1}{2} h^2$$

is a supersolution, and $\gamma(x,h) \geq Q(x) + c_0$. Now, the solution of the obstacle problem has the property of being the minimal supersolution lying above the obstacle. Hence, $\gamma(x,h) \geq P_0(x)$. Dividing by h^2 we get $D_{ee}P_0 \geq -1$ for any direction e. Now, $D_{ee}P_0$ is a supersolution outside the coincidence set, so $D_{ee}P_0 > -1$ there. \square

Lemma 6.5 On each component of the set $\{\widetilde{u} > 0\}$ we have

$$\widetilde{p} = Q + c$$

for some constant (depending possibly on the component). In fact, it holds on the closure of each component. We also have $\widetilde{p} \to 0$ as $|x| \to \infty$.

The proof is easy using the previous arguments.

Lemma 6.6 If P_0 is the corresponding solution of the obstacle problem for data $\Phi = Q + c_0$, then $P_0 \geq \widetilde{p}$ and $P_0 = \widetilde{p}$ in the coincidence set of P_0 .

Proof. Let $P_t = P_0 + t$ so that $P_t \to t > 0$ as $|x| \to \infty$. Then, $P_t - Q$ is strictly convex outside the coincidence set (see proof above), thus the gradient never vanishes. The obstacle is now $\Phi_t = \Phi + t$ so there is no coincidence between P_t and $\Phi = Q(x) + c_0$. Let Λ_0 be the coincidence set of P_0 , the closure of a ball $B_\rho(0)$, and let Θ_0 be the connected component of $\{\widetilde{p} = Q + c_0\}$ near the origin.

From the previous estimates, for t large P_t does not touch \widetilde{p} . Let us now decrease t and suppose that there is a first $t_0 > 0$ such that P_{t_0} touches \widetilde{p} .

The possibility of first contact $P_t = \tilde{p}$ at a point x_0 such that $\tilde{u}(x_0) = 0$ contradicts the maximum principle, unless P_{t_0} and \tilde{p} coincide and then $t_0 = 0$.

The first contact cannot happen either at a point x_0 where $(-\Delta)^s P_0 = 0$ and $\widetilde{u} > 0$ since then $\widetilde{p} = Q + c_1$ $(c_1 > c_0)$ in a neighborhood of x_0 , and we have already proved that ∇P_{t_0} cannot equal ∇Q .

Finally, we have to consider the case $(-\Delta)^s P_0 > 0$ and $\widetilde{u} > 0$, where $P_t = Q + c_0 + t$ while $\widetilde{p} = Q + c$. This forces again $t_0 = 0$. This concludes that proof that $P_0 \geq \widetilde{p}$.

In order to check that $\Lambda_0 \subset \Theta_0$ we consider for contradiction the existence of a point $x_0 \in B_{\rho}(0)$ such that $x_0 \in \partial \Theta_0$. In that case $P_0 - \widetilde{p}$ has a zero minimum there with $(-\Delta)^s P_0(x_0) > 0$, $(-\Delta)^s \widetilde{p}(x_0) = 0$, hence $(-\Delta)^s (P_0 - \widetilde{p})(x_0) > 0$ which contradicts the maximum principle. \square

Lemma 6.7 We have $\widetilde{p} \geq Q + c_0$.

Proof. Suppose that $\widetilde{p} - Q - c_0$ has a negative minimum at some point x_0 . We can translate \widetilde{p} up and bring it down until it touches P_0 by above. This must happen on the set where $u_0 = 0$, since $\Lambda_0 \subset \Theta_0$. But this contradicts the maximum principle since $\widetilde{u} \geq 0$ at x_0 and P_0 and \widetilde{p} are not equal.

Finally, we prove that \widetilde{p} is a supersolution and $\widetilde{p} \geq P_0$ (the least supersolution). \square

7 End of proof of Theorem 4.1

We now go back to the end of Section 5 and revert to the notations y and v. Once we have uniquely identified the pressure p_* at a.e. time as the solution P_0 of the obstacle problem for a certain constant c_0 that might depend on time, the scaling laws (cf. (3.12)) allow to uniquely identify c_0 as the constant such that $\int (-\Delta^s P_0)(y) dy = M$. Hence, c_0 is fixed in time and we conclude by continuity that $p_*(y,\tau) = P_{c_0}(y)$.

The above argument also identifies in a unique way the constant asymptotic limit $v_*(y,\tau) = V_C(y)$. It follows that the convergence $v_j \to V_c$ as $\tau_j \to \infty$ takes place independently of the subsequence $\tau_j \to \infty$, and we conclude that $v_j(\cdot,\tau) \to V_C$ as $\tau \to \infty$. Since the families v_j are compact in $L^{\infty}_{loc}(\mathbb{R}^n)$ the convergence is uniform in that space. On the other hand, from the conservation of mass and the uniform bound for $\int |y|^2 v(y,\tau) dy$ we conclude that the convergence takes place also in $L^1(\mathbb{R}^n)$.

We still need to check the uniform convergence of $v(\cdot, \tau)$ to V_C in the whole space. This is a rather simple calculus lemma using the following facts:

- (i) for $\tau \geq 1$ the funtions $v(\cdot, \tau)$ are uniformly bounded,
- (ii) for every $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that $\int_{|y| \ge R(\varepsilon)} v(y, \tau) dy < \varepsilon$ uniformly in τ .
- (iii) uniformly bounded functions are C^{α} continuous with a uniform bound.

This ends the proof for v. The translation of the results in terms of the variable u is immediate.

8 The question of a spectral gap

When using the entropy—entropy dissipation approach, we are interested in the following inequality

(8.1)
$$\mathcal{E}(v) \le C \, \mathcal{I}(v)$$

for a class \mathcal{C} of functions that includes our solutions and for a constant C > 0 which is called the spectral gap. In our case it means that for all $v \in \mathcal{C}$ we must have

(8.2)
$$\int_{\mathbb{R}^n} |\mathcal{H}v|^2 dy + \beta \int_{\mathbb{R}^n} |y|^2 v \, dy \le C \int \left| \nabla (\mathcal{K}v + \frac{\beta}{2} |y|^2) \right|^2 v dy.$$

Developing the last integral leads to

$$\int |\mathcal{H}v|^2 dy + \beta \int |y|^2 v dy \le C \int |\nabla \mathcal{K}v|^2 v dy + C\beta^2 \int |y|^2 v dy + 2C\beta \int (\nabla \mathcal{K}v.y) v dy.$$

We can transform the last integral into

$$\int (\nabla \mathcal{K}v.y) \, v dy = -n \int \mathcal{K}v \, v dy - \int (\mathcal{K}v)(\nabla v.y) \, dy.$$

For many solutions the last term will be positive, hence we need

$$(1 + 2Cn\beta) \int |\mathcal{H}v|^2 dy \le C \int |\nabla \mathcal{K}v|^2 v dy + (C\beta^2 - \beta) \int |y|^2 v dy.$$

The question is: can we prove this strange interpolation inequality? Since the order of differentiation in the first term is -s, for the second 1-2s, while the third is zero. It is clear that the case s = 1/2 is different and maybe simpler.

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References

- [1] I. Athanasopoulos and L. A. Caffarelli. Optimal regularity of lower dimensional obstacle problems. Preprint.
- [2] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa. The structure of the free boundary for lower dimensional obstacle problems. Preprint, arXiv.math.
- [3] P. Biler, C. Imbert, G. Karch, Fractal porous medium equation, Preprint.
- [4] P. Biler, G. Karch, and R. Monneau. Nonlinear diffusion of dislocation density and self-similar solutions. Comm. Math. Phys. (2009), online.
- [5] L. A. Caffarelli. The obstacle problem revisited, The Journal of Fourier Analysis and Applications 4 (4-5): 383–402,
- [6] L. A. Caffarelli. Further regularity for the Signorini problem. Comm. Partial Differential Equations 4 (1979), no. 9, 1067–1075.
- [7] L. A. Caffarelli, S. Salsa, and L. Silvestre. Regularity estimates for the solution and the free boundary to the obstacle problem for the fractional Laplacian, arXiv:math.AP/0702392v1, 2007.
- [8] L. A. Caffarelli and L. Silvestre. An extension problem related to the fractional laplacian, Comm. Partial Diff. Eqns., to appear.

- [9] L. A. Caffarelli, F. Soria, and J. L. Vazquez. Regularity of solutions of the fractional porous medium flow, in preparation.
- [10] L. A. Caffarelli and A. Vasseur. Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation. Preprint
- [11] L. A. Caffarelli and J. L. Vázquez, Nonlinear porous medium flow with fractional potential pressure, arXiv:1001.0410 [math.AP]
- [12] A. Friedman, Variational principles and free boundary problems, New York: Wiley, (1982).
- [13] D. Gilbarg and N. S. Trudinger. "Elliptic partial differential equations of second order". Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin, 2001.
- [14] A. K. Head. Dislocation group dynamics II. Similarity solutions of the continuum approximation. Phil. Mag. **26** (1972), 65–72.
- [15] N. S. Landkof. "Foundations of modern potential theory". Die Grundlehren der mathematischen Wissenschaften, Band 180. Translated from the Russian by A. P. Doohovskoy. Springer, New York, 1972.
- [16] L. E. Silvestre. Hölder estimates for solutions of integro differential equations like the fractional Laplace.
- [17] E. M. Stein. "Singular integrals and differentiability properties of functions". Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
- [18] J. L. Vázquez. "The porous medium equation. Mathematical theory", Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford (2007).
- [19] Vázquez, J. L. "Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type". Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006. ISBN: 978-0-19-920297-3; 0-19-920297-4.

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